Yang's model of charged particle scattering with energy loss

G. Sandison, L. Papież, and V. Tulovsky
Department of Radiation Oncology, Indiana University, 535 Barnhill Drive, Indianapolis, Indiana 46202
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Yang's model [Phys. Rev. Lett. **84**, 599 (1951)] for the scattering of charged particles assumes a constant energy for the penetrating particles. This constraint has been removed in this work by allowing the loss of particle energy to be proportional to the depth of penetration. Exact solutions are found in the form of convergent series for cases I and II defined by Yang. The consistency of these solutions with Yang's solutions for constant energy is verified.

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I. INTRODUCTION

In a review of cosmic ray theory, Rossi and Greisen [1] cite Fermi as deriving a simple transport equation describing the penetration of charged particles through a scattering medium, and providing a complete solution for the particle distribution. This derivation assumed small-angle scattering with randomly distributed scattering centers and a constant energy for the penetrating particles. Eyges [2] removed this constant energy constraint by deriving a solution to the transport equation which allowed the particle energy to be a function of the penetration depth. Then Yang [3], while again assuming a constant energy for the penetrating charged particles, improved upon the earlier work of Fermi by accounting for the difference between the *depth* achieved by a particle and the actual *path length* it traveled. This difference was coined the "excess path length." A general solution for the joint distribution function of angle, position, and excess path length was not provided by Yang [3]. However, conditional solutions for two cases of interest, termed case I and case II, were given by him. Case I describes the excess path length distribution at a given depth of *all* electrons, while case II describes the excess path length distribution at a given depth for electrons traveling perpendicularly to the surface of the scattering medium. In a parallel development Spencer and Coyne [4] derived approximate solutions to the Lewis equation [5], and under the conditions of small-angle scattering obtained expressions analogous to the first term of the Yang series solutions but with energy loss included. Finally, Nakatsuka [6], again assuming a constant energy for the penetrating charged particle, provided an approximate solution of the Yang transport equation.

This paper provides exact solutions to the Yang transport equation [3] for his cases I and II when the particle energy is allowed to be a linear function of the penetration depth.

II. DESCRIPTION OF THE PROBLEM

In the classical theory of angular scattering for charged particles originally presented by Fermi [1,2], the assumption of small cumulative change of angle is imposed, and the angular process is modeled by a random walk on the plane. In the limit of infinitesimally small steps and an infinitely large number of collisions (i.e., in the diffusional limit) this angular process is equivalent to a two-dimensional Brownian motion on the plane. This means, in particular, that the projected angles of travel Θ_x and Θ_y are independent, one-dimensional Brownian motions. The processes Θ_x and Θ_y in turn determine the stochastic evolutions of the lateral coordinates x and y of the scattered particle, as well as its projected excess path lengths ε_x and ε_y. To determine these evolutions one only needs to use the geometrical formulas [1,3] which relate variable Θ_x with x and ε_x, and variable Θ_y with y and ε_y, respectively. The standard description of the evolution of processes Θ_x, Θ_y, x, y, ε_x, and ε_y with depth z is given by their joint probability density distribution function $F(z,x,y,\Theta_x,\Theta_y,\varepsilon_x,\varepsilon_y)$. Due to the independence of θ_x and θ_y the function $F(z,x,y,\Theta_x,\Theta_y,\varepsilon_x,\varepsilon_y)$ is separable into the product

$$F(z,x,y,\Theta_x,\Theta_y,\varepsilon_x,\varepsilon_y)=F(z,x,\Theta_x,\varepsilon_x)F(z,y,\Theta_y,\varepsilon_y) ,$$

where $F(z,x,\Theta_x,\varepsilon_x)$ and $F(z,y,\Theta_y,\varepsilon_y)$ denote the joint probability density functions of the projected motions of a particle on the (z,x) and (z,y) planes, respectively. Thus, without loss of generality, we can investigate the transport equation for processes Θ_x, x, and ε_x alone, and write [3]

$$\frac{\partial F}{\partial z}=-\Theta_x\frac{\partial F}{\partial x}+\frac{k}{4}\frac{\partial^2 F}{\partial \Theta_x^2}-\tfrac{1}{2}\Theta_x^2\frac{\partial F}{\partial \varepsilon_x} , \tag{1a}$$

where F is a function of z, x, Θ_x, and ε_x and parameter k is the linear angular scattering power [1,7]. In the nomenclature of stochastic process theory, the parameter k has the meaning of one half the value of the diffusion constant for the process $\Theta_x(z)$. In Yang's original paper, k was assumed to be a constant. In general, k depends upon the atomic number of the scattering medium and the energy of the penetrating particle as follows:

$$k(E)=k_0\left[\frac{m_0c^2(E+m_0c^2)}{E(E+2m_0c^2)}\right]^2 , \tag{1b}$$

where m_0c^2 stands for the rest mass energy (0.511 MeV) of the electron, e is the electronic charge, and the scattering constant k_0 is given by

$$k_0=\frac{16\pi e^4}{(m_0c^2)^2}\sum_i N_iZ_i(Z_i+1)\ln(204Z_i^{-1/3}) ,$$

where N_i is the number of atoms per unit volume of element Z_i in the scattering medium. For broad electron beams incident upon water, the most probable energy loss is linearly dependent on depth [8], and so we can replace E in (1b) by

$$E=E_0(1-z/R_p) ,$$

where R_p is the practical range of the incident electrons in the beam, and E_0 is the incident electron energy. The high-energy approximation ($E \gg m_0c^2$) allows us to write (1b) as

$$k(z)=k_0\left[\frac{m_0c^2}{E_0}\right]^2\frac{1}{(1-z/R_p)^2} . \tag{1c}$$

Assuming the same initial condition for the solution of equation (1a) as Yang[3],

$$F(0,x,\Theta_x,\varepsilon_x)=\delta(x)\delta(\Theta_x)\delta(\varepsilon_x) \tag{2}$$

and taking into account the fact that $F(z,x,\Theta_x,0)=0$ (i.e., the probability measure equals 0 on the set of identically equal to zero trajectories of the diffusion process $\Theta_x(z)$ on $[0,z)$), we find that the Laplace transform $\bar{F}_\mu$ of the solution of (1a) and (2) with respect to the projected excess path length variable ε_x satisfies the equation

$$\frac{\partial\bar{F}_\mu}{\partial z}=-\Theta_x\frac{\partial\bar{F}_\mu}{\partial x}+\frac{k(z)}{4}\frac{\partial^2\bar{F}_\mu}{\partial\Theta_x^2}-\frac{\mu}{2}\Theta_x^2\bar{F}_\mu , \tag{3a}$$

with the initial condition

$$\bar{F}_\mu(0,x,\Theta_x)=\delta(x)\delta(\Theta_x) . \tag{3b}$$

The Laplace transform function $\bar{F}_\mu(z,x,\Theta_x)$ is defined explicitly by

$$\begin{aligned}\bar{F}_\mu(z,x,\Theta_x)&=\int_0^\infty F(z,x,\Theta_x,\varepsilon_x)e^{-\mu\varepsilon_x}d\varepsilon_x\\&=\mathcal{L}[F(z,x,\Theta_x,\varepsilon_x)] ,\end{aligned} \tag{4}$$

where $\mathcal{L}$ represents the Laplace transformation with respect to ε_x. For any fixed z, x, and Θ_x the function $F(z,x,\Theta_x,\varepsilon_x)$ is a probability density distribution of the random variable ε_x and thus belongs to the class of integrable functions $L^1([0,\infty))$. Consequently, $\bar{F}_\mu(z,x,\Theta_x)$ defined by (4) is the analytic function in the half-plane $\mathrm{Re}\mu>0$.

III. SOLUTION

Equation (3a) is of the same form as a transport equation previously used by Papież and Sandison [9] to describe charged particle penetration in dense media. However, in Ref. [9] the notation $\gamma/2$ was used in place of the constant μ, and γ had the interpretation of the positive constant (or positive function of z). To obtain the solution for (3a) and (3b), where μ is a complex parameter with $\mathrm{Re}\mu>0$, we first allow $\gamma/2$ in formula (4.14) of Ref. [9] to be any positive parameter, and then analytically continue the function (4.14) of Ref. [9] to the half-plane $\mathrm{Re}\gamma>0$. This analytic continuation is unique and amounts to the substitution of $\gamma(z)/2$ in the general formula (4.14) of Ref. [9] by the complex parameter μ, $\mathrm{Re}\mu>0$. The resulting function is then the exact solution of (3a), with μ being a complex parameter, such that $\mathrm{Re}\mu>0$.

Case I: Excess path-length density distribution for all particles at a given depth

We denote the excess path length density distribution of all particles at depth z as $F(z,\varepsilon)$. Due to the small-angle approximation the excess path length ε can be written as $\varepsilon=\varepsilon_x+\varepsilon_y$, with ε_x and ε_y being independent random variables. Thus $F(z,\varepsilon)$ can be expressed as

$$\begin{aligned}F(z,\varepsilon)&=\int F(z,\varepsilon_x,\varepsilon-\varepsilon_x)d\varepsilon_x\\&=\int F(z,\varepsilon_x)F(z,\varepsilon-\varepsilon_x)d\varepsilon_x\\&=F(z,\varepsilon_x)\circ F(z,\varepsilon_y) ,\end{aligned} \tag{5}$$

where

$$F(z,\varepsilon_x)=\int\int F(z,x,\Theta_x,\varepsilon_x)dx\,d\Theta_x , \tag{6}$$

$$F(z,\varepsilon_x,\varepsilon_y)=\int\int\int\int F(z,x,y,\Theta_x,\Theta_y,\varepsilon_x,\varepsilon_y)\times d\Theta_x d\Theta_y dx\,dy \tag{7}$$

and $F(z,\varepsilon_x)\circ F(z,\varepsilon_y)$ denotes the convolution of the projected excess path-length distributions $F(z,\varepsilon_x)$ and $F(z,\varepsilon_y)$. Taking the Laplace transform with respect to ε of (5), we obtain

$$\begin{aligned}\mathcal{L}[F(z,\varepsilon)]&=\mathcal{L}[F(z,\varepsilon_x)]\mathcal{L}[F(z,\varepsilon_y)]\\&=\bar{F}_\mu(z)\bar{F}_\mu(z)=\bar{F}_\mu^2(z)\end{aligned} \tag{8}$$

or

$$F(z,\varepsilon)=\mathcal{L}^{-1}[\bar{F}_\mu^2(z)] , \tag{9}$$

where

$$\begin{aligned}\bar{F}_\mu(z)&=\mathcal{L}[\int\int F(z,x,\Theta_x,\varepsilon_x)dx\,d\Theta_x]\\&=\int\int\bar{F}_\mu(z,x,\Theta_x)dx\,d\Theta_x\end{aligned} \tag{10}$$

and $\mathcal{L}^{-1}$ denotes the inverse Laplace transform with respect to μ. Formulas (4.14) and (4.3a) of Ref. [9] show that $\bar{F}_\mu(z,x,\Theta_x)$ is a normalized two-dimensional Gaussian multiplied by a depth-dependent factor $C(z,\mu)$. Thus, integrating $\bar{F}_\mu(z,x,\Theta_x)$ over x and Θ_x, we find from formulas (10) and (9) that

$$\bar{F}_\mu(z)=C(z,\mu) , \tag{11}$$

and thus

$$F(z,\varepsilon)=\mathcal{L}^{-1}[C^2(z,\mu)] . \tag{12}$$

The expression for $C^2(z,\mu)$ is [see formulas (4.13c) and (4.14) of Ref. [9]]

$$C^2(z,\mu)=e^{-\mu\int_0^z h(z',\mu)dz'} , \tag{13}$$

where $h(z,\mu)$ is the solution to the first-order differential equation [formula (4.12a) of Ref. [9]]

$$\frac{dh(z,\mu)}{dz}+\mu h^2(z,\mu)=\frac{k(z)}{2} , \tag{14}$$

with initial condition $h(0,\mu)=0$.

An exact closed-form solution of the *Ricatti* equation (14) is possible for a function $k(z)$ given by (1c), and can be written as [10] (formula 1.143)

$$h(z,\mu)=\frac{b}{R_p(1-z/R_p)}\frac{1-(1-z/R_p)^{\sqrt{1+2\mu b}}}{(1+\sqrt{1+2\mu b})+(\sqrt{1+2\mu b}-1)(1-z/R_p)^{\sqrt{1+2\mu b}}}\ , \tag{15a}$$

where

$$b=k_0(m_0c^2)^2(R_p/E_0)^2\ . \tag{15b}$$

Substituting (15) into (13), we find that

$$C^2(z,\mu)=(1-z/R_p)^{-1/2}\frac{2\sqrt{1+2b\mu}(1-z/R_p)^{\sqrt{1+2b\mu}/2}}{(1+\sqrt{1+2b\mu})+(\sqrt{1+2b\mu}-1)(1-z/R_p)^{\sqrt{1+2b\mu}}}\ . \tag{16}$$

The expression (16), treated as a function of μ, is the analytic (in the right half-plane $\mathrm{Re}\mu>0$) function, which can be expanded in the series of functions of the type $\{e^{-\alpha_n(z)\mu}\mu^{-n},\alpha_n(z)$—given functions of z, $n=1,2,3,\ldots\}$. If then the inverse Laplace transform (12) is applied to this series [11], we obtain the following formula for $F(z,\varepsilon)$:

$$F(z,\varepsilon)=(1-z/R_p)^{-1/2}e^{-\varepsilon/2b}\frac{\hat{a}(z)}{\pi^{1/2}\varepsilon^{3/2}}\sum_{n=0}^{\infty}(-1)^n(2n+1)e^{-\hat{a}^2(z)(2n+1)^2/4\varepsilon}$$
$$+\frac{1}{2b\pi^{1/2}}\sum_{n=0}^{\infty}\left\{(1-z/R_p)^{-n-1}\int_{\{\sqrt{\varepsilon/2b}+[(2n+1)\hat{a}(z)]/2\sqrt{\varepsilon}\}}^{\infty}G\left[\frac{(2n+1)\hat{a}(z)}{\sqrt{2b}},\varepsilon,n,u\right]e^{-u^2}du\right\}\ , \tag{17a}$$

where $\hat{a}(z)$ and $G(c,\varepsilon,n,u)$ are defined by the following expressions:

$$\hat{a}(z)=\frac{|\ln(1-z/R_p)|\sqrt{b}}{\sqrt{2}}\ , \tag{17b}$$

$$G(c,\varepsilon,n,u)=\sum_{\nu=0}^{n}\frac{2b}{\varepsilon}\left\{\frac{n!(-1)^{n-\nu+1}2^{\nu+1}(2n-\nu+1)}{\nu!(\nu+1)!(n-\nu)!}\left[u\left[\frac{2\varepsilon}{b}\right]^{1/2}-\frac{\varepsilon}{b}\right]\left[u\left[\frac{2\varepsilon}{b}\right]^{1/2}-\frac{\varepsilon}{b}-c\right]^{\nu}\right\}\ . \tag{17c}$$

Case II: Excess path length of all particles traveling perpendicularly to the surface at a given depth

We denote the excess path-length density distribution of all particles traveling perpendicularly to the surface at depth z as $F(z,\varepsilon|\Theta=0)$. As in case I, we obtain the equality

$$F(z,\varepsilon|\Theta=0)=F(z,\varepsilon_x|\Theta_x=0)\circ F(z,\varepsilon_y|\Theta_y=0)\ , \tag{18}$$

where the right-hand side of (18) denotes the convolution of the projected excess path-length distributions $F(z,\varepsilon_x|\Theta_x=0)$ and $F(z,\varepsilon_y|\Theta_y=0)$, conditional on projected angles Θ_x and Θ_y being equal to zero at depth z, and Θ is the directional vector for the particle $[\Theta=(\Theta_x,\Theta_y)]$. By definition, the conditional probability density $F(z,\varepsilon_x|\Theta_x=0)$ can be written as

$$F(z,\varepsilon_x|\Theta_x=0)=\frac{F(z,\Theta_x=0,\varepsilon_x)}{F(z,\Theta_x=0)}\ , \tag{19}$$

where $F(z,\Theta_x=0,\varepsilon_x)$ is the function $F(z,x,\Theta_x,\varepsilon_x)$ integrated over x, with Θ_x set equal to 0, while $F(z,\Theta_x=0)$ is the function $F(z,x,\Theta_x,\varepsilon_x)$ integrated over x and ε_x with Θ_x set equal to 0. Analogous to (8) for case I, we obtain from (18) that

$$\begin{aligned}\mathcal{L}[F(z,\Theta=0,\varepsilon)]&=\mathcal{L}[F(z,\Theta_x=0,\varepsilon_x)]\mathcal{L}[F(z,\Theta_y=0,\varepsilon_y)]\\&=\bar{F}_\mu(z,\Theta_x=0)\bar{F}_\mu(z,\Theta_y=0)\\&=\bar{F}^2_\mu(z,\Theta_x=0)\end{aligned} \tag{20}$$

or

$$F(z,\Theta=0,\varepsilon)=\mathcal{L}^{-1}[\bar{F}^2_\mu(z,\Theta_x=0)]\ , \tag{21}$$

where

$$\bar{F}_\mu(z,\Theta_x=0)=\int\bar{F}_\mu(z,x,\Theta_x=0)dx\ . \tag{22}$$

The function $\bar{F}_\mu(z,x,\Theta_x)$ is a normalized Gaussian given by (4.14) of Ref. [9]. Integrating this function with respect to x and substituting $\Theta_x=0$ results in the expression

$$\bar{F}_\mu(z,\Theta_x=0)=\frac{C(z,\mu)}{\sqrt{2\pi h(z,\mu)}}\ . \tag{23}$$

Substituting (15) and (16) into (23), (21), and (19), and then performing the inverse Laplace transform, gives

$$F(z,\varepsilon|\theta=0)=\frac{2(z\sqrt{b}/R_p\sqrt{2})}{\pi^{1/2}\varepsilon^{5/2}}e^{-\varepsilon/2b}\times\sum_{n=0}^{\infty}\left[\frac{(2n+1)^2\hat{a}^2(z)}{4}-\frac{\varepsilon}{2}\right]\times e^{-[(2n+1)^2\hat{a}^2(z)/4\varepsilon]}\ , \tag{24}$$

where $\hat{a}(z)$ is given by (17b).

IV. CHECK OF CONSISTENCY AND EXAMPLES

To check the consistency of our results with the previous results of Yang, we have considered the case when $R_p\to\infty$. In this case the function $k(z)$ given by (1c) has the limit $k=k_0(m_0c^2/E_0)^2$, and we expect that our solution will coincide with Yang's solution for the constant k above.

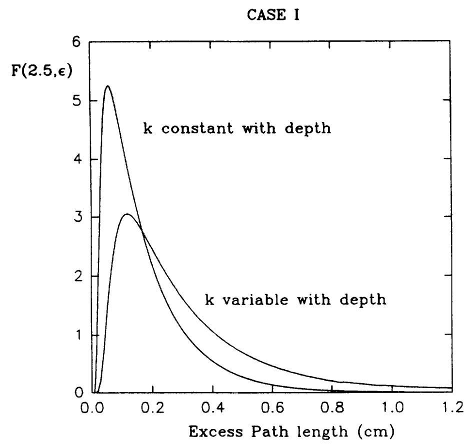

FIG. 1. Case I (all electrons): Excess path length distribution at a depth of 2.5 cm in water for a 10-MeV incident electron beam.

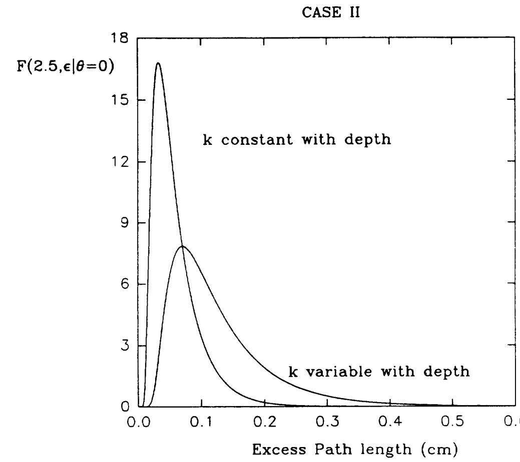

FIG. 2. Case II (electrons traveling perpendicularly to the surface): Excess path length distribution at a depth of 2.5 cm in water for a 10-MeV incident electron beam.

The limit of the function $\hat{a}(z)$ given by (17b), when $R_p \to \infty$ is equal to $a(z)=z\sqrt{k/2}$. In case I the limit of the first sum in (17a) gives us exactly the Yang solution

$$F(z,\varepsilon)=\frac{a(z)}{\pi^{1/2}\varepsilon^{3/2}}\sum_{n=0}^{\infty}(-1)^n(2n+1)\times\exp\left[-\frac{a^2(z)(2n+1)^2}{4\varepsilon}\right], \quad (25)$$

and the limit of the second sum in (17a) is equal to 0.

In case II the expression $z\sqrt{b}/(R_p\sqrt{2})$ in (24) is exactly $a(z)$, and as $R_p \to \infty$ the formula (24) gives us the sum

$$F(z,\varepsilon|\theta=0)=\frac{2a(z)}{\pi^{1/2}\varepsilon^{5/2}}\sum_{n=0}^{\infty}\left[\frac{(2n+1)^2a^2(z)}{4}-\frac{\varepsilon}{2}\right]\times\exp\left[-\frac{a^2(z)(2n+1)^2}{4\varepsilon}\right], \quad (26)$$

which is exactly the Yang solution.

To illustrate the effect of the energy dependence of the angular scattering power k on the excess path-length distributions $F(z,\varepsilon)$ and $F(z,\varepsilon|\Theta=0)$, let us consider the penetration of electrons in water, the usual dosimetric medium.

Results have been generated for a monoenergetic beam of 10-MeV electrons incident upon a homogeneous water medium at a depth of 2.5 cm, which is approximately equal to half the practical range. The values of the different constants in (1c) in this case are

$$k_0=47.4\ \text{cm}^{-1},\quad R_p=4.834\ \text{cm},\quad E_0=10\ \text{MeV}\ .$$

Figure 1 presents the variation of $F(2.5,\varepsilon)$, with ε (at $z=2.5$ cm) for the case when k is considered constant with depth, and for the case when k is considered dependent on z according to formula (1c). Notice that for k dependent on z, the probability of an electron having a larger excess path length at $z=2.5$ cm is increased. The most probable excess path length for the distribution also increases, i.e., for this example, from 0.06 cm when k is constant to 0.12 cm when k is dependent on z. The values of $F(2.5,\varepsilon)$ were computed using the first seven terms of the series (17) and (25), respectively. The value of total probability $\int_0^\infty F(2.5,\varepsilon)d\varepsilon$ was verified numerically as being equal to 1.000.

Figure 2 presents the variation of $F(2.5,\varepsilon|\Theta=0)$, with ε (at $z=2.5$ cm) for the case when k is considered constant with depth, and for the case when k is considered dependent on z according to formula (1c). Similarly to the data for $F(2.5,\varepsilon)$, the probability of an electron having a larger excess path length at $z=2.5$ cm is increased when k is considered dependent on z. The most probable excess path length for case II is increased in this example from 0.03 cm when k is constant, to 0.07 cm when k is dependent on z. The values of $F(2.5,\varepsilon|\Theta=0)$ for k constant were computed using the first six terms of the series (24) and (26), respectively. The value of the total probability $\int_0^\infty F(2.5,\varepsilon|\Theta=0)d\varepsilon$ was verified numerically as being equal to 1.000.

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